Complete System for a Collineation Group Isomorphic with the Group of the Double Tangents of a Plane Quartic.

Dissertation

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by

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Introduction.

The group of the double tangents of a plane quartic is isomorphic with one of a series of groups arising in connection with the theta functions. This one is associated with the division into half periods for $p=3$. Its immediate predecessor* associated analytically with the division into half periods for $p=2$ is the group of order $16 \cdot 720$ associated geometrically with the Kummer** surface.

* This group was discussed by Burkhardt who gave an historical account of the matter to the time of his papers (about 1890). They appeared in the Mathematische Annalen, vols. 35, 36, 38.

** An account of this group in relation to the Kummer surface is to be found in Hudson's "Kummer's Quartic Surface", which appeared in 1905

A similar one is determined by the division into thirds of periods for theta functions for $p = 2$, and is associated geometrically with the lines of a cubic surface. In all of these cases isomorphic collineation groups have been discovered and discussed in considerable detail but no collineation group isomorphic with the group of the double tangents has been discussed. It is the purpose of this paper to derive and discuss such a group. The group being connected with the quartic curve, by proper mapping methods a collineation group is obtained in which the variables are irrational invariants of the quartic itself. The equations of the quartic and its double tangents are obtained in a form whose symmetry

and simplicity leave nothing to be desired. A complete system for the collineation group and associated canonical forms of the quartic are obtained and discussed. The collineation group appears in seven variables. That this is the smallest number of variables in which this group can be represented as a collineation group is evident from a theorem in Weber's "Lehrbuch der Algebra", Vol. II., p. 376. Certain configurations of some interest appear. The results obtained are applicable to the solution of the equation of the double tangents of the quartic and should also be valuable for discussing certain invariants of the quartic and configurations of the double

tangents. The quartic appears with an isolated flex and may throw some light on the hitherto unsolved problem of the flexes.

I.

The Cremona Group $G_{7,2}$ of P_7^2 in S_6.

Two sets of seven points in a plane, P_7^2 and Q_7^2, ordered with respect to each other, are <u>congruent under the Cremona transformation</u> C_m with ρ F-points if ρ of the pairs p_i, q_i $(i=1,\cdots 7)$ are corresponding F-points of C_m, and if the remaining $7-\rho \geqq 0$ of the pairs p_i, q_i are pairs of ordinary corresponding points under C_m. The number of projectively distinct sets congruent to P_7^2 is the number of types of Cremona transformations. To determine this number the following theorem* is necessary:

<u>The general Cremona transformation C_m $(m>2)$ with ρ F-points is projectively</u>

* Transactions of the American Mathematical Society, Vol. XVII, p. 348

determined when there are given the order m, the ρ F-points, their multiplicities subject to the conditions $\sum_1^\rho r_i^2 = m^2-1$, $\sum_1^\rho r_i = 3(m-1)$, and the positions of four corresponding F-points.

The possible transformations to be considered in connection with P_7^2 are given by the following table where α_j is the number of F-points of multiplicity j:

	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{D}_4$	$\mathcal{C}_5$	$\mathcal{D}_5$	$\mathcal{D}_6$	$\mathcal{D}_7$	$\mathcal{D}_8$
α_1	3	4	3	6	0	3	1	0	0
α_2		1	3	0	6	3	4	3	0
α_3				1		1	2	4	7

$\mathcal{C}$ is used to indicate a transformation with 6 or fewer F-points, $\mathcal{D}$ one with 7 F-points. Using in addition to these transformations the collineation $\mathcal{C}_1$, we find the number of transformations

$C_1, C_2, C_3, C_4, C_5, D_4, D_5, D_6, D_7, D_8$ to be respectively $\binom{7}{0}, \binom{7}{3}, \binom{7}{4}\binom{3}{1}, \binom{7}{3}\binom{4}{3}, \binom{7}{6}, \binom{7}{6}, \binom{7}{3}\binom{4}{3}, \binom{7}{1}\binom{6}{4}, \binom{7}{3}, \binom{7}{0}$ or 2.288. But since P_7^2 and Q_7^2 congruent under D_8 are projective, there are only 288 projectively distinct types of congruence.

The point sets P_7^2 are mapped* upon the points of a space S_6 by taking P_7^2 in the canonical form

$$P_7^2 : \quad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ x_1 & y_1 & u \\ x_2 & y_2 & u \\ x_3 & y_3 & u \end{matrix}$$

and regarding $x_1, x_2, x_3, y_1, y_2, y_3, u$ as the coordinates of a point in S_6. Then

* Trans. Am. Math. Soc., Vol. XIII, pp. 353-4

if two sets of points are congruent under a Cremona transformation in S_2 their maps in S_6 are corresponding points under a Cremona transformation in S_6. The general Cremona transformation in S_2 can be expressed as a product of quadratic factors. The corresponding effect on the map of P_7^2 in S_6 is that of an involutory Cremona transformation. Moreover any Cremona transformation of the kind considered is a product of transformations corresponding to quadratic transformations in S_2. Any quadratic transformation in S_2 with F-points at points of P_7^2 can be obtained from a single one by permutation of the points. Hence $G_{7,2}$, the Cremona group of P_7^2 in S_6, can be generated by the symmetric group of order 7!

and a single transformation in S_6 corresponding to a quadratic transformation in S_2. The number of operations in $G_{7,2}$ is clearly the same as the number of types of congruences of sets P_7^2 if further it is required that P_7^2 be ordered. $G_{7,2}$ is thus seen to be of order $7!\,288$.

II.

Point Sets on a Cuspidal Cubic.

The cuspidal cubic curve $C_1 \equiv x_2^3 - x_1 x_3^2 = 0$ is given parametrically by
$x_1 = t^3$; $x_2 = t$; $x_3 = 1$; the parameter of the cusp being $t = \infty$ and that of the flex, $t = 0$. Hence, given C_1, a set of seven points P_7^2 is determined by seven parameters t_i $(i = 1, 2, \cdots 7)$. If on the other hand, seven

points only are given they determine a net of cubics containing among them 24 cuspidal cubics. Thus the fact that C_1 is given is equivalent to the assumption of a single solution of the cusp equation of degree twenty-four of the net. P_7^2 determined in this way is general.

The condition that t_i and t_j coincide is

(1) $$t_i - t_j = 0$$

The condition that three points t_i be on a line is

(2) $$\sum^{3} t_i = 0$$

The condition that six points t_i be on a conic is

(3) $$\sum^{6} t_i = 0$$

The quadratic transformation A_{123} with F-points at t_1, t_2, and t_3 sends C_1

into another cuspidal cubic C_1', whose points can be named by means of the same parameter t. C_1' can be sent by a collineation into C_1. This operation sends a point t on C_1' into a point t' of C_1. To determine the effect upon the parameters we note that if

(4) $$t_i' = t_i + \tfrac{1}{3}(t_1 + t_2 + t_3) \text{ then}$$

$$t_i' + t_j' + t_k' = t_i + t_j + t_k + t_1 + t_2 + t_3$$

That is, the requirement that to three points on a line correspond three points on a conic through the F-points is satisfied. This gives the effect of the transformation upon an ordinary point. It is clear that the condition that a point coincide with an F-point goes into the condition that the corresponding point be on the opposite F-line. Hence

$$t_i' - t_3' = t_i + t_1 + t_2$$

and by means of (4) we obtain the relation

$$t_3' = t_3 - \tfrac{2}{3}(t_1 + t_2 + t_3)$$

The effect of A_{123} is that of the collineation on the parameters, given by the equations

$$
\begin{aligned}
A_{123}:\quad t_1' &= t_1 - \tfrac{2}{3}(t_1 + t_2 + t_3)\\
t_2' &= t_2 - \tfrac{2}{3}(t_1 + t_2 + t_3)\\
t_3' &= t_3 - \tfrac{2}{3}(t_1 + t_2 + t_3)\\
t_i' &= t_i + \tfrac{1}{3}(t_1 + t_2 + t_3) \qquad i = 4, 5, 6, 7.
\end{aligned}
$$

where t_1, t_2, and t_3 are F-points and t_i ordinary points.

The aggregate of operations obtained by taking products of A_{123} and permutations of $t_1, t_2, \text{------} t_7$ constitute the group $T_{7,2}$ of P_7^2 on C_1. An element of $T_{7,2}$ can be looked upon as the operation of passing from one P_7^2 on C_1 to a congruent one named by seven

other values t_i'. We get in this way 288 <u>projectively distinct</u> sets of points on $\mathcal{C}$, congruent in some order. Hence there are $7!\,288$ projectively distinct ordered sets. $T_{7,2}$, the collineation group on the variables $t_1, t_2, \ldots t_7$ is of order $7!\,288$.

III.

Invariants of $T_{7,2}$.

An invariant of $T_{7,2}$ is a function of the t's unaltered by the operations of $T_{7,2}$. The condition that two points coincide is sent by A_{123} into the condition that two points coincide or that three points be on a line; the condition that three points be on a line is sent into the condition that two points

coincide, that three points be on a line, or that six points be on a conic; the condition that six points be on a conic is sent into the condition that six points be on a conic or that three points be on a line. The algebraic expressions, (1), (2), and (3) of II, for these conditions are permuted as stated but may change sign. Hence

$$I_2 = \sum_{21}(t_1 - t_2)^2 + \sum_{35}(t_1 + t_2 + t_3)^2 + \sum_{7}(t_1 + t_2 + t_3 + t_4 + t_5 + t_6)^2$$
$$= 9(3a_1^2 - 4a_2),$$

where a_i is a symmetric function of the t's of degree i, is an invariant of $\Gamma_{7,2}$ of degree 2. Likewise are found

$$I_4 = \sum_{21}(t_1 - t_2)^4 + \sum_{35}(t_1 + t_2 + t_3)^4 + \sum_{7}(a_1 - t_1)^4$$
$$= 3(3a_1^2 - 4a_2)^2$$

$$I_6 = \sum_{21}(t_1 - t_2)^6 + \sum_{35}(t_1 + t_2 + t_3)^6 + \sum_{7}(a_1 - t_1)^6$$
$$= 27a_1^6 - 108a_1^4a_2 + 192a_1^2a_2^2 - 96a_2^3 - 72a_1^3a_3$$
$$+ 24a_1a_2a_3 - 36a_3^2 + 72a_1^2a_4 + 48a_2a_4 - 72a_1a_5 - 288a_6$$

which are invariants of degrees 4 and 6 respectively. I_4 is seen to be a multiple of I_2^2.

IV

The Quartic C^4 arising from a Set of Seven Points.

The plane E_x of P_7^2 is mapped upon a plane E_Y by the cubic curves on P_7^2. To the cubics of E_x correspond the lines of E_Y. Hence to two points of E_x corresponds one point of E_Y. If however a double point of a curve of the net on P_7^2 is taken, it alone corresponds to a point of E_Y, for the two variable intersections of curves of the net have coincided. Since the Jacobian is the locus of double points of the net, the correspondence between the Jacobian

of the net on P_7^2 and its map in E_γ is one to one. The Jacobian of the net on P_7^2 has double points at the points of P_7^2 and being of order 6 will have $6 \times 3 - 7 \times 2 = 4$ variable intersections with curves of the net. That is, the map of the Jacobian squared, since pairs have coincided on it, is a quartic curve C^4 in E_γ. The cubics of the net with a double point map into the lines of C^4, but the 21 degenerate P_{ij}, consisting of the line $t_i t_j$ and the conic on the remaining five points, and the 7 cubics P_{0i}, with a double point at a point of P_7^2, map into the 28 double tangents of C^4 in such a way that the 7 P_{0i} give rise to an Aronhold set. The 24 cuspidal cubics of the net map into the flex tangents and the 24 cusps into the flexes of C^4.

The operations of $T_{7,2}$ transform P_7^2 into Q_7^2 and transform the net of curves on P_7^2 into a net on Q_7^2. The curves P_{0i} and P_{ij} of the net on P_7^2 are transformed into the curves Q_{0i} and Q_{ij} of Q_7^2. The effect of A_{123} on the curves P_{0i} and P_{ij} is that of the permutation

$$(P_{01}\,P_{23})(P_{02}\,P_{31})(P_{03}\,P_{12})(P_{45}\,P_{67})(P_{46}\,P_{57})(P_{47}\,P_{56})$$

Hence, the effect of the product $A_{237}\,A_{457}\,A_{167}$ is that of the interchange of subscripts 0 and 7. $G_{7!}$ together with the transposition (07) generates a subgroup of $T_{7,2}$ the symmetric group $G_{8!}$ of the permutations of 0, 1, 2, --- 7. By comparing the notation above for the curves P with the Hesse notation $[ik;\ i,k = 1, 2, \cdots 8;\ i \neq k]$ of the double tangents of the quartic it is seen at once that $G_{8!}$ and A_{123} effect the same permutations on the curves P

as the subgroup* E and the substitution P_{1238}, which generate the group of the double tangents, effect on the double tangents. Since the order of $T_{7,2}$ is that of the group of the double tangents of the quartic,

$T_{7,2}$ is simply isomorphic with the group of the double tangents of the quartic.

V.

A Net of Cubics on P_7^2.

We will obtain the quartic map of the Jacobian for the net of cubics on $t_1, t_2, \ldots t_7$ formed by taking the three following base cubics

* Finite Groups, Miller, Blichfeldt and Dickson, pp. 362–365.

$\mathcal{C}_1 \equiv -x_1 x_3^2 + x_2^3 = 0$, the cuspidal cubic above;

$$\mathcal{C}_2 \equiv x_1^2 x_2 - a_1 x_1^2 x_3 + a_2 x_1 x_2^2 - a_3 x_1 x_2 x_3 + a_4 x_2^3 - a_5 x_2^2 x_3 + a_6 x_2 x_3^2 - a_7 x_3^3 = 0,$$

the cubic on $t_1, t_2, \cdots t_7$ having no term in $x_1 x_3^2$ and passing through the cusp of $\mathcal{C}_1$;

$$\mathcal{C}_3 \equiv 4x_1^3 + x_1^2(c_2 x_2 - c_3 x_3) + x_1(c_4 x_2^2 - c_5 x_2 x_3) + c_6 x_2^3 - c_7 x_2^2 x_3 + c_8 x_2 x_3^2 - c_9 x_3^3 = 0,$$

the cubic on $t_1, t_2, \cdots t_7$, having no term in $x_1 x_3^2$ and touching $\mathcal{C}_1$ at $t = -\frac{1}{2}a_1$.

The a_i are symmetric functions of $t_1, t_2, \cdots t_7$ of degree i; the c's are given in terms of the a's by

$$c_n = 4a_n - 4a_1 a_{n-1} + a_1^2 a_{n-2}$$

VI.

The Jacobian $J[\mathcal{C}_1 \mathcal{C}_2 \mathcal{C}_3]$ and its Map $\mathcal{C}^4$.

We have now to determine the Jacobian J of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ and obtain the

quartic map of J^2 by means of the equations

$$x_1' = C_1 \;,\; x_2' = C_2 \;,\; x_3' = C_3$$

$$J \equiv \begin{vmatrix} -x_3^2 & 3x_2^2 & -2x_1x_3 \\ (2x_1x_2 - 2ax_1x_3 + a_2x_2^2 - a_3x_2x_3) & (x_1^2 + 2a_2x_1x_2 - a_3x_1x_3 + 3a_4x_2^2 - 2a_5x_2x_3 + a_6x_3^2) & (-a_1x_1^2 - a_3x_1x_2 - a_5x_2^2 + 2a_6x_2x_3 - 3a_7x_3^2) \\ (12x_1^2 + 2c_2x_1x_2 - 2c_3x_1x_3 + c_4x_2^2 - c_5x_2x_3) & (c_2x_1^2 + 2c_4x_1x_2 - c_5x_1x_3 + 3c_6x_2^2 - 2c_7x_2x_3 + c_8x_3^2) & (-c_3x_1^2 - c_5x_1x_2 - c_7x_2^2 + 2c_8x_2x_3 - 3c_9x_3^2) \end{vmatrix} \equiv$$

$$24x_1^5x_3 - 36a_1x_1^4x_2^2 + 48a_2x_1^4x_2x_3 + (-24a_3 + 3a_1c_2 - 3c_3)x_1^4x_3^2$$
$$+ (-36a_3 - 6a_1c_2 + 6c_3)x_1^3x_2^3 + (72a_4 + 6a_2c_2 - 6c_4)x_1^3x_2^2x_3$$
$$+ (-48a_5 - 3a_3c_2 - 6a_2c_3 + 6a_1c_4 + 3c_5)x_1^3x_2x_3^2$$
$$+ (24a_6 + 3a_3c_3 - 3a_1c_5)x_1^3x_3^3 + (-36a_5 - 6a_3c_2 + 3a_2c_3 - 3a_1c_4 + 6c_5)x_1^2x_2^4$$
$$+ (72a_6 + 12a_4c_2 + 3a_3c_3 - 3a_1c_5 - 12c_6)x_1^2x_2^3x_3$$
$$+ (-108a_7 - 9a_5c_2 - 9a_4c_3 + 9a_1c_6 + 9c_7)x_1^2x_2^2x_3^2$$
$$+ (6a_6c_2 + 6a_5c_3 - 6a_1c_7 - 6c_8)x_1^2x_2x_3^3 + (-3a_7c_2 - 3a_6c_3 + 3a_1c_8 + 3c_9)x_1^2x_3^4$$
$$+ (-6a_5c_2 - 3a_3c_4 + 3a_2c_5 + 6c_7)x_1x_2^5$$
$$+ (12a_6c_2 + 6a_5c_3 + 6a_4c_4 - 6a_2c_6 - 6a_1c_7 - 12c_8)x_1x_2^4x_3$$
$$+ (-18a_7c_2 - 12a_6c_3 - 6a_5c_4 - 3a_4c_5 + 3a_3c_6 + 6a_2c_7 + 12a_1c_8 + 18c_9)x_1x_2^3x_3^2$$
$$+ (18a_7c_3 + 6a_6c_4 + 3a_5c_5 - 3a_3c_7 - 6a_2c_8 - 18a_1c_9)x_1x_2^2x_3^3$$

$$+(-6a_7c_4-3a_6c_5+3a_3c_8+6a_2c_9)x_1x_2x_3^4+(3a_7c_5-3a_3c_9)x_1x_3^5$$

$$+(-3a_5c_4+3a_2c_7)x_2^6+(3a_6c_4+3a_5c_5-3a_3c_7-6a_2c_8)x_2^5x_3$$

$$+(-9a_7c_4-6a_6c_5-3a_5c_6+3a_4c_7+6a_3c_8+9a_2c_9)x_2^4x_3^2$$

$$+(9a_7c_5+6a_6c_6-6a_4c_8-9a_3c_9)x_2^3x_3^3$$

$$+(-9a_7c_6-3a_6c_7+3a_5c_8+9a_4c_9)x_2^2x_3^4+(6a_7c_7-6a_5c_9)x_1x_3^5$$

$$+(-3a_7c_8+3a_6c_9)x_3^6=0$$

$\left(\frac{J}{3}\right)^2$ expressed in terms of the cubics $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, i.e. the map $\mathcal{C}^4$ in E_Y if the $\mathcal{C}$'s are regarded as reference lines is

$$\left[\frac{J}{3}\right]^2 \equiv (64a_1a_2a_3a_5a_7-32a_2a_3a_6a_7-64a_1a_3^2a_4a_7$$
$$+32a_1a_2^2a_6a_7-64a_1^2a_2^2a_5a_7+64a_1^2a_2a_3a_4a_7-64a_4a_7^2$$
$$-64a_1a_5^2a_7+128a_1^2a_4a_5a_7-64a_1^3a_4^2a_7+64a_5a_6a_7$$
$$-64a_1a_4a_6a_7+16a_2^2a_7^2+16a_3^2a_6^2+16a_1^2a_2^2a_6^2$$
$$-32a_1a_2a_3a_6^2)\mathcal{C}_1^4+(8a_1^2a_3^2a_6-8a_1^3a_2a_3a_6+80a_1^3a_4a_7$$
$$-72a_1^2a_2a_3a_7+16a_1^3a_2^2a_7+16a_1^2a_6^2-16a_1^3a_5a_6+16a_1^4a_4a_6$$
$$-112a_1^2a_5a_7+64a_1a_3^2a_7+64a_3a_4a_7-32a_1a_2a_4a_7+64a_7^2$$
$$+64a_1a_6a_7+32a_1a_3a_4a_6-32a_1^2a_2a_4a_6-32a_2a_5^2$$

$$
\begin{aligned}
&-32a_3a_5a_6+32a_1a_2a_5a_6)\,\mathcal{C}_1^{3}\mathcal{C}_2+(8a_1a_2a_3a_6-8a_3^2a_6\\
&+48a_1a_4a_7+8a_2a_3a_7-16a_1a_2^2a_7-16a_6^2+16a_1a_5a_6\\
&-16a_1^2a_4a_6-16a_5a_7)\,\mathcal{C}_1^3\mathcal{C}_3+(16a_2a_6-16a_1a_2a_5\\
&+8a_1^3a_5-2a_1^2a_3^2+8a_1a_3a_4+8a_3a_5-48a_1a_7)\,\mathcal{C}_1^2\mathcal{C}_2\mathcal{C}_3\\
&+(a_1^4a_3^2-4a_1^5a_5-8a_1^4a_6+16a_1^2a_2a_6+16a_1^3a_2a_5\\
&-8a_1^3a_3a_4-8a_1^2a_3a_5-16a_1^3a_7+32a_1a_2a_7-32a_1a_3a_6\\
&+16a_5^2-32a_1a_4a_5+16a_1^2a_4^2-64a_3a_7)\,\mathcal{C}_1^2\mathcal{C}_2^2\\
&+(a_3^2-4a_1a_5+8a_6)\,\mathcal{C}_1^2\mathcal{C}_3^2+(a_1^6-4a_1^4a_2+8a_1^3a_3\\
&-16a_1^2a_4+32a_1a_5)\,\mathcal{C}_1\mathcal{C}_2^3+(8a_1^2a_2-3a_1^4-8a_1a_3\\
&-16a_4)\,\mathcal{C}_1\mathcal{C}_2^2\mathcal{C}_3+(3a_1^2-4a_2)\,\mathcal{C}_1\mathcal{C}_2\mathcal{C}_3^2-\mathcal{C}_1\mathcal{C}_3^3\\
&+16\,\mathcal{C}_2^3\mathcal{C}_3=0
\end{aligned}
$$

The quartic $\mathcal{C}^4$ is only projectively determined since the net of cubics is only projectively determined by the choice of seven points, that is, $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ can be any linearly independent cubics of the net. Moreover any transformation sending the net of cubics into a net of

cubics transforms C^4 point by point into itself. Any such transformation can be built up of factors like A_{123}. Suppose A_{123} is the transformation $x_i = q(x')$. Then $C_i(x)$ goes into $C_i[q(x')] \equiv C_i'(x')$. A point of C^4 is given by $y_i = C_i(x)$ where x is a point of $J[C_1 C_2 C_3]$. But the transform of the point is $y_i' = C_i'(x')$. Since $C_i'(x') \equiv C_i[q(x')] \equiv C_i(x)$, $y_i' = y_i$ and the point is unaltered. The curves P_{0i} and P_{ij} have however been permuted and the coefficients of C^4 are those derived from the transformed Aronhold set and are in general altered in form. For if the transformation A_{123} is applied to the set of points $t_1, t_2, \text{----} t_7$ and to the cubics C_1, C_2, C_3, we obtain a new set of points $t_1', t_2', \text{----} t_7'$ and three new cubics C_1', C_2', C_3' whose coefficients contain not only the

symmetric functions of the points $t_1', t_2', \text{----} t_7'$ but in addition the F points t_1', t_2', and t_3'. Likewise the transform of C^4 by A_{123}, that is, the map of $J^2[C_1' C_2' C_3']$ by C_1', C_2', and C_3', will contain the F points t_1', t_2', and t_3' besides the symmetric functions of $t_1', t_2', \text{----} t_7'$. If, however, C_1, C_2, and C_3 were cubics covariant under A_{123}, since J is covariant the coefficients of the quartic C^4 would be invariants of A_{123}, and since they are symmetric would be <u>invariants of $T_{1,2}$</u>. The cubics C_1, C_2, and C_3 are mapped into the reference lines of the plane E_Y. Since C_1 is covariant and is mapped into the known flex tangent, a triangle of reference determined uniquely by the flex would arise from a set of cubics covariant with the cuspidal cubic.

The problem of finding the above mentioned invariants of $T_{1,2}$ is then reduced to that of finding the coefficients of C^4 referred to a triangle of reference covariant with the flex.

A simple way to determine such a covariant triangle is to take

1) for x_1, the line C_1, the flex tangent;

2) for x_3, the tangent to C^4 at the intersection of C_1 with C^4 other than the flex $(0,0,1)$;

3) for x_2 the line joining $0,0,1$ with the intersection of x_3 with the polar line of $0,0,1$ as to the polar conic of $0,1,0$ as to C^4.

The above choice of reference lines gives the following linear transformation of the C's to the new variables x:

$$\mathcal{C}_1 = x_1$$

(5) $$\mathcal{C}_2 = \lambda x_1 + x_2$$

$$\mathcal{C}_3 = \mu x_1 + x_3$$

where $48\lambda = 3a_1^4 - 8a_1^2a_2 + 8a_1a_3 + 16a_4$

$$16\mu = -a_1^6 + 4a_1^4a_2 - 8a_1^3a_3 + 16a_1^2a_4 - 32a_1a_5$$

The following expressions A_i, invariants of degree i of $T_{7,2}$, are such numerical multiples of the coefficients of the transform of $\mathcal{C}^4$ as to remove fractional coefficients. α_{ijkl} is the coefficient of $x_i x_j x_k x_l$ where i, j, k, and l are 1, 2, or 3.

$$A_2 = \alpha_{1233} = 3a_1^2 - 4a_2$$

$$A_6 = 48\alpha_{1133} = 18a_1^6 - 72a_1^4a_2 + 96a_1^3a_3 + 32a_1^2a_2^2 - 96a_1^2a_4 - 32a_1a_2a_3 + 96a_1a_5 - 64a_2a_4 + 48a_3^2 + 384a_6$$

By comparing A_2 and A_6 with I_2

and I_6 obtained in III by symmetrizing it is seen that

$$9A_2 = I_2 \;,$$

and $6A_2^3 - 3A_6 = 4I_6$

$$A_8 = 48\alpha_{1123} = -27a_1^8 + 144a_1^6a_2 - 192a_1^5a_3$$
$$-160a_1^4a_2^2 + 192a_1^4a_4 + 320a_1^3a_2a_3$$
$$-192a_1^3a_5 - 128a_1^2a_2a_4 - 160a_1^2a_3^2$$
$$+128a_1a_3a_4 - 2304a_1a_7 + 768a_2a_6$$
$$+384a_3a_5 - 256a_4^2$$

$$A_{10} = 16\alpha_{1122} = 3a_1^{10} - 20a_1^8a_2 + 32a_1^7a_3 + 32a_1^6a_2^2$$
$$-32a_1^6a_4 - 96a_1^5a_2a_3 + 32a_1^5a_5 + 64a_1^4a_2a_4$$
$$+80a_1^4a_3^2 - 128a_1^4a_6 - 128a_1^3a_3a_4$$
$$-256a_1^3a_7 + 256a_1^2a_2a_6 + 128a_1^2a_3a_5$$
$$+512a_1a_2a_7 - 512a_1a_3a_6 - 1024a_3a_7$$
$$+256a_5^2.$$

$$A_{12} = 6912\alpha_{1113} = -117a_1^{12} + 936a_1^{10}a_2 - 1152a_1^{9}a_3$$

$$-2352a_1^{8}a_2^{2} + 5376a_1^{7}a_2a_3 + 4608a_1^{7}a_5$$

$$+1536a_1^{6}a_2a_4 - 2816a_1^{6}a_3^{2} - 6912a_1^{6}a_6$$

$$-5376a_1^{5}a_2^{2}a_3 - 8064a_1^{5}a_2a_5 + 1152a_1^{5}a_3a_4$$

$$-20736a_1^{5}a_7 - 4608a_1^{4}a_2^{2}a_4 + 8064a_1^{4}a_2a_3^{2}$$

$$+34560a_1^{4}a_2a_6 + 23040a_1^{4}a_3a_5 - 2304a_1^{4}a_4^{2}$$

$$+12288a_1^{3}a_2^{2}a_5 + 2304a_1^{3}a_2a_3^{2} + 3072a_1^{3}a_2a_3a_4$$

$$+55296a_1^{3}a_2a_7 - 10240a_1^{3}a_3^{3} - 55296a_1^{3}a_3a_6$$

$$-36864a_1^{3}a_4a_5 - 18432a_1^{2}a_2^{2}a_6 - 21504a_1^{2}a_2a_3a_5$$

$$+6144a_1^{2}a_2a_4^{2} + 12288a_1^{2}a_3^{2}a_4 - 55296a_1^{2}a_3a_7$$

$$+101376a_1^{2}a_5^{2} - 110592a_1a_2^{2}a_7$$

$$+73728a_1a_2a_3a_6 - 24576a_1a_2a_4a_5$$

$$-18432a_1a_3^{2}a_5 + 6144a_1a_3a_4^{2} + 221184a_1a_4a_7$$

$$-110592a_1a_5a_6 + 55296a_2a_3a_7$$

$$+36864a_2a_4a_6 - 55296a_3^{2}a_6 + 13824a_3a_4a_5$$

$$-8192a_4^{3} - 110592a_5a_7 - 110592a_6^{2}$$

$$
\begin{aligned}
A_{14} = 768\alpha_{1112} = {} & 27a_1^{14} - 252a_1^{12}a_2 + 384a_1^{11}a_3 \\
& + 752a_1^{10}a_2^2 - 384a_1^{10}a_4 - 2176a_1^9a_2a_3 + 384a_1^9a_5 \\
& - 608a_1^8a_2^3 + 1792a_1^8a_2a_4 + 1568a_1^8a_3^2 \\
& - 768a_1^8a_6 + 2816a_1^7a_2^2a_3 - 1280a_1^7a_2a_5 \\
& - 2944a_1^7a_3a_4 + 768a_1^7a_7 - 1536a_1^6a_2^2a_4 \\
& - 3968a_1^6a_2a_3^2 + 2816a_1^6a_2a_6 + 2432a_1^6a_3a_5 \\
& + 1280a_1^6a_4^2 + 4608a_1^5a_2a_3a_4 - 2048a_1^5a_2a_7 \\
& + 2048a_1^5a_3^3 - 5120a_1^5a_3a_6 - 2048a_1^5a_4a_5 \\
& - 1024a_1^4a_2^2a_6 - 512a_1^4a_2a_3a_5 - 1024a_1^4a_2a_4^2 \\
& - 4096a_1^4a_3^2a_4 + 8192a_1^4a_3a_7 + 8192a_1^4a_4a_6 \\
& - 1536a_1^4a_5^2 + 4096a_1^3a_2^2a_7 + 2048a_1^3a_3^2a_5 \\
& + 2048a_1^3a_3a_4^2 + 16384a_1^3a_4a_7 - 12288a_1^3a_5a_6 \\
& - 30720a_1^2a_2a_3a_7 - 4096a_1^2a_2a_4a_6 \\
& + 8192a_1^2a_2a_5^2 - 2048a_1^2a_3^2a_6 - 2048a_1^2a_3a_4a_5 \\
& - 12288a_1^2a_5a_7 + 12288a_1^2a_6^2 - 8192a_1a_2a_4a_7 \\
& + 32768a_1a_3^2a_7 + 8192a_1a_3a_4a_6 \\
& - 8192a_1a_3a_5^2 + 49152a_1a_6a_7 - 24576a_2a_5a_7 \\
& + 16384a_3a_4a_7 - 24576a_3a_5a_6 \\
& + 8192a_4a_5^2 + 49152a_7^2
\end{aligned}
$$

$$A_{18} = 9.\overline{16}^{3} a_{1111} = 63 a_1^{18} - 756 a_1^{16} a_2 + 1152 a_1^{15} a_3$$
$$+ 3264 a_1^{14} a_2^2 - 1152 a_1^{14} a_4 - 9600 a_1^{13} a_2 a_3 + 1728 a_1^{13} a_5$$
$$- 5600 a_1^{12} a_2^3 + 8448 a_1^{12} a_2 a_4 + 7056 a_1^{12} a_3^2$$
$$- 1152 a_1^{12} a_6 + 24576 a_1^{11} a_2^2 a_3 - 12288 a_1^{11} a_2 a_5$$
$$- 13824 a_1^{11} a_3 a_4 + 4608 a_1^{11} a_7 + 2816 a_1^{10} a_2^4$$
$$- 16896 a_1^{10} a_2^2 a_4 - 34944 a_1^{10} a_2 a_3^2 + 6144 a_1^{10} a_2 a_6$$
$$+ 21504 a_1^{10} a_3 a_5 + 6912 a_1^{10} a_4^2 - 17152 a_1^{9} a_2^3 a_3$$
$$+ 22016 a_1^{9} a_2^2 a_5 + 53760 a_1^{9} a_2 a_3 a_4 - 29184 a_1^{9} a_2 a_7$$
$$+ 17408 a_1^{9} a_3^3 - 10752 a_1^{9} a_3 a_6 - 30720 a_1^{9} a_4 a_5$$
$$+ 6656 a_1^{8} a_2^3 a_4 + 34816 a_1^{8} a_2^2 a_3^2 - 5120 a_1^{8} a_2^2 a_6$$
$$- 76288 a_1^{8} a_2 a_3 a_5 - 21504 a_1^{8} a_2 a_4^2 - 46080 a_1^{8} a_3^2 a_4$$
$$+ 52224 a_1^{8} a_3 a_7 + 61440 a_1^{8} a_4 a_6 + 20736 a_1^{8} a_5^2$$
$$- 20480 a_1^{7} a_2^2 a_3 a_4 + 106496 a_1^{7} a_2^2 a_7 - 31744 a_1^{7} a_2 a_3^3$$
$$- 8192 a_1^{7} a_2 a_3 a_6 + 114688 a_1^{7} a_2 a_4 a_5 + 72704 a_1^{7} a_3^2 a_5$$
$$+ 36864 a_1^{7} a_3 a_4^2 - 24576 a_1^{7} a_4 a_7 - 73728 a_1^{7} a_5 a_6$$
$$- 8192 a_1^{6} a_2^3 a_6 - 4096 a_1^{6} a_2^2 a_3 a_5 - 8192 a_1^{6} a_2^2 a_4^2$$
$$+ 30720 a_1^{6} a_2 a_3^2 a_4 - 299008 a_1^{6} a_2 a_3 a_7$$
$$- 237568 a_1^{6} a_2 a_4 a_6 - 61440 a_1^{6} a_2 a_5^2 + 11264 a_1^{6} a_3^4$$
$$+ 4096 a_1^{6} a_3^2 a_6 - 225280 a_1^{6} a_3 a_4 a_5 - 8192 a_1^{6} a_4^3$$

$$+73728a_1^6a_6^2-212992a_1^5a_2^3a_7+131072a_1^5a_2^2a_3a_6$$

$$-4096a_1^5a_2a_3^2a_5+24576a_1^5a_2a_3a_4^2+139264a_1^5a_2a_4a_7$$

$$+24576a_1^5a_2a_5a_6-12288a_1^5a_3^3a_4+229376a_1^5a_3^2a_7$$

$$+385024a_1^5a_3a_4a_6+172032a_1^5a_3a_5^2+180224a_1^5a_4^2a_5$$

$$+147456a_1^5a_6a_7+778240a_1^4a_2^2a_3a_7+81920a_1^4a_2^2a_4a_6$$

$$-81920a_1^4a_2^2a_5^2-286720a_1^4a_2a_3^2a_6+40960a_1^4a_2a_3a_4a_5$$

$$-32768a_1^4a_2a_4^3-122880a_1^4a_2a_5a_7-245760a_1^4a_2a_6^2$$

$$+8192a_1^4a_3^3a_5-36864a_1^4a_3^2a_4^2-409600a_1^4a_3a_4a_7$$

$$-466944a_1^4a_3a_5a_6-425984a_1^4a_4^2a_6-466944a_1^4a_4a_5^2$$

$$+147456a_1^4a_7^2-327680a_1^3a_2^2a_4a_7-819200a_1^3a_2a_3^2a_7$$

$$+163840a_1^3a_2a_3a_5^2-393216a_1^3a_2a_6a_7$$

$$+163840a_1^3a_3^3a_6-16384a_1^3a_3^2a_4a_5+65536a_1^3a_3a_4^3$$

$$+196608a_1^3a_3a_5a_7+393216a_1^3a_3a_6^2-262144a_1^3a_4^2a_7$$

$$+1572864a_1^3a_4a_5a_6+294912a_1^3a_5^3-983040a_1^2a_2^2a_5a_7$$

$$+589824a_1^2a_2^2a_6^2+1572864a_1^2a_2a_3a_4a_7$$

$$-393216a_1^2a_2a_3a_5a_6-131072a_1^2a_2a_4^2a_6$$

$$+131072a_1^2a_2a_4a_5^2-393216a_1^2a_2a_7^2+327680a_1^2a_3^3a_7$$

$$-131072a_1^2a_3^2a_4a_6-81920a_1^2a_3^2a_5^2$$

$$-65536a_1^2a_3a_4^2a_5+393216a_1^2a_3a_6a_7$$

$$
\begin{aligned}
&+393216\,a_1^2a_4a_5a_7-393216\,a_1^2a_4a_6^2-1179648\,a_1^2a_5^2a_6\\
&+1179648\,a_1a_2^2a_6a_7-1179648\,a_1a_2a_3a_6^2\\
&-262144\,a_1a_2a_4^2a_7+393216\,a_1a_3^2a_5a_6\\
&+262144\,a_1a_3a_4^2a_6-131072\,a_1a_3a_4a_5^2\\
&+393216\,a_1a_3a_7^2-1572864\,a_1a_4a_6a_7-1179648\,a_1a_5^2a_7\\
&+1179648\,a_1a_5a_6^2+589824\,a_2^2a_7^2-1179648\,a_2a_3a_6a_7\\
&-393216\,a_2a_4a_5a_7+589824\,a_3^2a_6^2+524288\,a_3a_4^2a_7\\
&-393216\,a_3a_4a_5a_6+65536\,a_4^2a_5^2-1572864\,a_4a_7^2\\
&+2359296\,a_5a_6a_7
\end{aligned}
$$

The Equation of C^4 Referred to the Chosen Reference Lines Covariant with the Flex.

$$
\begin{aligned}
&3A_{18}x_1^4+144A_{14}x_1^3x_2+16A_{12}x_1^3x_3+6912A_{10}x_1^2x_2^2\\
&+2304A_8x_1^2x_2x_3+2304A_6x_1^2x_3^2+110592A_2x_1x_2x_3^2\\
&-110592x_1x_3^3+1769472x_2^3x_3=0
\end{aligned}
$$

VII.

Double Tangents of C^4

The double tangents of C^4 in E_y are of two types.

1) The type (oi) are the maps of the cubic curves P_{oi} of the net in E_x with a double point at t_i. There are 7 of this type and they form an Aronhold set.

2) The type (ij) are the maps of the cubic curves P_{ij} of the net in E_x consisting of the line $t_i t_j$ and the conic on the remaining five points. There are 21 of this type.

To determine the equation of the double tangent (oi) the equation of P_{oi} must be found. The curves of the net having a common tangent at t_i

form a pencil whose equation is:

$$\kappa(x_2^3 - x_1x_3^2) + x_1^3 + (a_2 - a_1^2 - a_1t_i - t_i^2)x_1^2x_2$$
$$-(a_3 - a_1a_2 - a_1^2t_i - a_1t_i^2)x_1^2x_3 + (a_4 - a_1a_3 - a_1a_2t_i - a_2t_i^2)x_1x_2^2$$
$$-(a_5 - a_1a_4 - a_1a_3t_i - a_3t_i^2)x_1x_2x_3 + (a_6 - a_1a_5 - a_1a_4t_i - a_4t_i^2)x_1x_3^2$$
$$-(a_7 - a_1a_6 - a_1a_5t_i - a_5t_i^2)x_2^2x_3 + (-a_1a_7 - a_1a_6t_i - a_6t_i^2)x_2x_3^2$$
$$-(-a_1a_7t_i - a_7t_i^2)x_3^3 = 0$$

There is a single member of this pencil with a double point at t_i. This is the curve for which κ has such a value that t_i satisfies a derivative of the pencil. This value of κ is

$$\kappa = t_i^6 + a_2t_i^4 + (a_1a_2 - a_3)t_i^3 - a_5t_i + (a_6 - a_1a_5)$$

The map in E_γ of this equation for the value of κ above, that is, the equation of the double tangent (oi) is

$$4[t_i^6 + a_2t_i^4 + (a_1a_2 - a_3)t_i^3 - a_5t_i + a_6 - a_1a_5]\xi_1 - (a_1 + 2t_i)^2\xi_2 + \xi_3 = 0$$

The equation of the line t_it_j is

$$x_1 + (s_2 - s_1^2)x_2 + s_1s_2x_3 = 0$$

where s_k is the symmetric function of t_i and t_j of degree k. The equation of the conic on the remaining five points is

$$x_1^2+(\sigma_4-\sigma_1\sigma_3)x_2^2-\sigma_1\sigma_5 x_3^2+(\sigma_2-\sigma_1^2)x_1x_2-(\sigma_3-\sigma_1\sigma_2)x_1x_3$$
$$-(\sigma_5-\sigma_1\sigma_4)x_2x_3=0$$

where σ_k is the symmetric function of the five points of degree k.

To obtain the equation of the double tangent (ij) in E_Y we have to find the map of the product of the equations of the line and conic above. This product, expressed in terms of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, that is, the equation of the double tangent (ij), is after removing numerical fractions

$$4(\sigma_1\sigma_5+s_1s_2\sigma_3-s_1s_2\sigma_1\sigma_2)\mathcal{C}_1-(a_1-2s_1)^2\mathcal{C}_2+\mathcal{C}_3=0$$

VIII

Proof of the Completeness of the System of Invariants.

The determination of the t's depends on the separation of the double tangents of the quartic and the isolation of a single flex. The t's are then projective irrational invariants of the quartic. Any function of the t's of proper weight is therefore a projective invariant of the quartic. Hence any invariant of $T_{1,2}$ is an irrational invariant of the quartic such that the only irrationality present is that of the flex. Such an invariant is expressible rationally and integrally in terms of the coefficients of the quartic and of the coordinates of the isolated flex. But

since the quartic has for coefficients the invariants $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18} and the coordinates of the flex are 0,0,1, every invariant of $T_{7,2}$ is rationally and integrally expressible in terms of the invariants $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18}. Moreover it is obvious when special values are given to $a_1, a_2, \cdots a_7$ that no one of the invariants A_i can be expressed rationally and integrally in terms of the others. Hence none of them is superfluous and

The invariants $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18} form a complete system for the group $T_{7,2}$.

IX

The Jacobian of $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18}

If an expression is alternating under the operations of $'\Gamma_{7,2}$, it contains as a factor $t_i - t_j$ and all its conjugate values under the operations of $\Gamma_{7,2}$. Hence it has as a factor

$$J \equiv \prod^{28}(t_i - t_j)\prod^{35}(t_i + t_j + t_k)\prod^{7}(a_1 - t_i)$$

which is an alternating expression of degree 63.

The Jacobian of A_2 $A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18} is an alternating expression of degree 63 and is therefore to within a numerical factor the product J. Since J at most changes sign when the operations of $\Gamma_{7,2}$ are carried out its square is an invariant of degree 126 and is rationally expressible in terms of the

invariants $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}$, and A_{18}.

X.

The Group $\Gamma'_{8,3}$ of P_8^3, a Set of Base Points of a Net of Quadrics.

There are in general an infinite number of sets of eight points in space congruent to a single set P_8^3 under a Cremona transformation which can be decomposed into a product of cubic Cremona transformations with F-points at points of P_8^3. If, however, P_8^3 is a set of base points of a net of quadrics we can make use of the following theorem*:

<u>If P_8^3 is a set of base points of a net of quadrics there are only 36 projectively</u>

* Trans. Amer. Math. Soc., Vol. XXII, p. 377 (45)

distinct sets congruent in some order to P_8^3.

There are then 36 types of congruence if no account is taken of the order of the points. If we require that P_8^3 be an ordered set we have, since P_8^3 can be ordered in 8! ways, 8! 36 types of congruence. The aggregate of operations transforming P_8^3 into the 8! 36 congruent sets constitutes a group which we will call $T_{8,3}$. Any one of these operations is, as presupposed, the product of cubic transformations which can be obtained from a single one by a permutation of the points of P_8^3. Hence $T_{8,3}$ is generated by a cubic transformation and the symmetric group of permutations of the points of P_8^3 of order 8! Abstractly then $T_{8,3}$ has as generators precisely the set to which the generators of $T_{7,2}$ were shown to be

equivalent in IV. $T_{8,3}$ and $T_{7,2}$ are therefore abstractly the same groups.

XI.

P_8^3 on a Cuspidal Quartic.

A cuspidal quartic $\mathfrak{D}$ in space is determined by the parametric equations

$$\mathfrak{D}: \quad x_1 = t_4, \quad x_2 = t^2, \quad x_3 = t, \quad x_4 = 1$$

The condition that two points t_i and t_j coincide is

$$t_i - t_j = 0$$

The condition that four points be in a plane is

$$\sum^4 t_i = 0$$

The condition that eight points be on a quadric is

$$\sum^8 t_i = 0$$

XII.

A Net of Quadrics on P_8^3.
Generators of the Group $T_{8,3}$.

Since we wish to consider P_8^3 as the base points of a net of quadrics, we determine P_8^3 by a choice of eight values t_i subject to the single condition $\sum^8 t_i = 0$, for the quartic above is the intersection of the quadrics

$$Q_2 \equiv x_1x_4 - x_2^2 = 0 \quad \text{and} \quad Q_3 \equiv x_2x_4 - x_3^2 = 0.$$

A third quadric on P_8^3 is

$$Q_1 \equiv x_1^2 + b_2x_1x_2 - b_3x_1x_3 + b_4x_2^2 - b_5x_2x_3 + b_6x_3^2 - b_7x_3x_4 + b_8x_4^2 = 0$$

where b_i is the symmetric function of the t's of degree i.

<u>P_8^3 determined by the choice of 8 t's subject to the single condition $b_1 = 0$ is the set of base points of the net of quadrics</u>

$$y_1Q_1 + y_2Q_2 + y_3Q_3 = 0$$

Generators of the group $T_{8,3}$ of P_8^3 determined in this way consist of the symmetric group of permutations of the 8 t's, together with a transformation on the t's corresponding to a cubic Cremona transformation A_{1234} with F-points at points of P_8^3, say at t_1, t_2, t_3, t_4; for the effect of A_{1234} is to send the net of quadrics into a net of quadrics and the cuspidal quartic $\mathfrak{D}$ above into another cuspidal quartic $\mathfrak{D}'$ whose points are named by the same parameter t. The quartic $\mathfrak{D}'$ can be sent back into $\mathfrak{D}$ by means of a collineation carrying the point t of $\mathfrak{D}'$ into the point t' of $\mathfrak{D}$. Thus can the transformation A_{1234} be regarded as a transformation upon the parameters t to new parameters t'.

The transformation

$$t_1' = t_1 - \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4)$$
$$t_2' = t_2 - \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4)$$
$$t_3' = t_3 - \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4)$$
$$t_4' = t_4 - \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4)$$
$$t_i' = t_i + \tfrac{1}{2}(t_1 + t_2 + t_3 + t_4) \qquad i = 5, 6, 7, 8.$$

is seen to have the effect of permuting the conditions that two points coincide, that four points be in a plane and that the eight points be on a quadric as does the transformation A_{1234} and thus gives the effect of A_{1234} upon P_8^3 in terms of the parameters t_i.

XIII

The Quartic D^4

If we consider y_1, y_2, y_3 as the coördinates of a point in a plane we have

by means of the net of quadrics

$$y_1 Q_1 + y_2 Q_2 + y_3 Q_3 = 0$$

a correspondence between the points of a plane and the quadrics of the net. To a pencil of quadrics on the elliptic quartic curve carrying the pencil corresponds a line of the plane. Corresponding to the quadrics of the net with a double point we will have a certain locus in the plane. Since in each pencil of the net are four quadrics with a double point, the locus is a quartic curve. Its equation in variables y_i found by writing the discriminant of the net is

$$D^4 \equiv \begin{vmatrix} 2y_1 & b_2 y_1 & -b_3 y_1 & y_2 \\ b_2 y_1 & 2(b_4 y_1 - y_2) & -b_5 y_1 & y_3 \\ -b_3 y_1 & -b_5 y_1 & 2(b_6 y_1 - y_3) & -b_7 y_1 \\ y_2 & y_3 & -b_7 y_1 & 2 b_8 y_1 \end{vmatrix} \equiv$$

$$
\begin{aligned}
&(-4b_2^2b_6b_8+b_2^2b_7^2+4b_2b_3b_5b_8-4b_3^2b_4b_8+16b_4b_6b_8-4b_4b_7^2)y_1^4\\
&+(-2b_2b_5b_7+4b_3^2b_8+4b_3b_4b_7-16b_6b_8+4b_7^2)y_1^3y_2\\
&+(4b_2^2b_8-2b_2b_3b_7-16b_4b_8+4b_5b_7)y_1^3y_3\\
&+(-4b_3b_7-4b_4b_6+b_5^2)y_1^2y_2^2+(4b_2b_6-2b_3b_5+16b_8)y_1^2y_2y_3\\
&+(b_3^2-4b_6)y_1^2y_3^2+4b_6y_1y_2^3+4b_4y_1y_2^2y_3\\
&-4b_2y_1y_2y_3^2+4y_1y_3^3-4y_2^3y_3=
\end{aligned}
$$

XIV.

Complete System for $T_{8,3}$.

The quartic D^4 consists of precisely the same terms as does C^4 above. The flex tangent $y_1=0$ now is the map of the quartic D which is unaltered by the operations of $T_{8,3}$. If we choose as base quadrics of the net quadrics covariant with D, since the discriminant is an invariant of the net its coefficients will then be invariants of $T_{8,3}$.

Since the quartic $\mathfrak{D}$ maps into the flex tangent, we need not determine these quadrics but have only to choose in the plane a triangle of reference covariant with the flex. The lines are chosen in the same way as for $\mathfrak{C}^4$. The transformation is therefore of the same type as (5), removes the same terms from $\mathfrak{D}^4$ as (5) does from $\mathfrak{C}^4$, and is

$$y_1 = 3y_1'$$
$$y_2 = b_4 y_1' + 3y_2'$$
$$y_3 = 3b_6 y_1' + 3y_3'$$

The transformed expression for $\mathfrak{D}^4$ is

$$3B_{18} y_1^4 + 27B_{14} y_1^3 y_2 + 3B_{12} y_1^3 y_3 + 81B_{10} y_1^2 y_2^2 + 27B_8 y_1^2 y_2 y_3 + 27B_6 y_1^2 y_3^2 + 81B_2 y_1 y_2 y_3^2 + 324 y_1 y_3^3 - 324 y_2^3 y_3 = 0$$

where B_i is an invariant of $T_{8,3}$ of degree i, whose explicit expressions are as follows

$$\mathcal{B}_2 = -4\,b_2$$

$$\mathcal{B}_6 = 3\,b_3^2 - 4\,b_2 b_4 + 24\,b_6$$

$$\mathcal{B}_8 = 4\,b_4^2 - 12\,b_2 b_6 - 6\,b_3 b_5 + 48\,b_8$$

$$\mathcal{B}_{10} = -4\,b_3 b_7 + b_5^2$$

$$\mathcal{B}_{12} = 54\,b_3^2 b_6 - 18\,b_3 b_4 b_5 + 8\,b_4^3 + 108\,b_6^2$$

$$\mathcal{B}_{14} = -6\,b_2 b_5 b_7 + 12\,b_3^2 b_8 + 4\,b_3 b_4 b_7 + 2\,b_4 b_5^2 + 12\,b_7^2$$

$$\begin{aligned}\mathcal{B}_{18} = {} & 27\,b_2^2 b_7^2 + 108\,b_2 b_3 b_5 b_8 - 54\,b_2 b_3 b_6 b_7 \\ & -18\,b_2 b_4 b_5 b_7 - 12\,b_3^2 b_4 b_8 + 27\,b_3^2 b_6^2 \\ & +24\,b_3 b_4^2 b_7 - 18\,b_3 b_4 b_5 b_6 + 3\,b_4^2 b_5^2 - 72\,b_4 b_7^2 \\ & -108\,b_5^2 b_8 + 108\,b_5 b_6 b_8\end{aligned}$$

It is to be noted that great simplicity is gained in the complete system when the group is represented in eight variables whose sum is zero. This is to be expected since every term in which b_1 enters vanishes. Moreover the notation for the double tangents is symmetrical.

The lines of D^4 arise from the

pencils of quadrics of the net. The quadrics in a pencil with a double point correspond to the contacts of the line with $\mathfrak{D}^4$. To a pencil of quadrics such that the double points have coincided in pairs corresponds a double tangent of $\mathfrak{D}^4$. Such a pencil of the net can be found by requiring that it contain the line $t_i t_j$. To do this we have only to require that the quadric contain a point of the line $t_i t_j$ other than t_i and t_j. That is, the linear condition on the y's is

$$(ij): (\sigma_6 + s_2\sigma_4 + s_1 s_2 \sigma_3 + s_1^2 s_2 \sigma_2) y_1 - s_1^2 y_2 - y_3 = 0$$

where s_k is the symmetric function of t_i and t_j of degree i and σ_k the symmetric function of the remaining six t's. This is the condition in the plane that the point y be on a double tangent. In other words it is the equation of the double tangent. The twenty-eight double tangents of $\mathfrak{D}^4$ are all of the type (ij) and are thus accounted for by the twenty-eight equations (ij).

Biographical Note.

Charles Clinton Bramble was born at Centreville, Md., Aug. 17, 1890. After graduation from the Centreville High School he entered Dickinson College and was awarded the degrees of Ph. B in 1912 and A. M. in 1913. During the academic year 1912-13 he was instructor in the Montclair Academy, Montclair, N. J. The following two years he pursued graduate courses at Johns Hopkins University and held during these years a Hopkins Scholarship. The year 1915-16 he was Lecturer in Mathematics at Bryn Mawr College. The year 1916-17 he continued his studies at Johns Hopkins as Fellow in Mathematics.

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